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Linear Birth and Death Models and Associated Laguerre and Meixner Polynomials

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We study birth and death processes with linear rates $\lambda_n = n + \alpha + c + 1$, $\mu_{n+1} = n + c$, $n \geqslant 0$ and μ_0 is either zero or c. The spectral measures of both processes are found using generating functions and the integral transforms of Laplace and Stieltjes. The corresponding orthogonal polynomials generalize Laguerre polynomials and the choice $\mu_0 = c$ generates the associated Laguerre polynomials of Askey and Wimp. We investigate the orthogonal polynomials in both cases and give alternate proofs of some of the results of Askey and Wimp on the associated Laguerre polynomials. We also identify the spectra of the associated Charlier and Meixner polynomials as zeros of certain transcendental equations.

1. INTRODUCTION

A birth and death process is a stationary Markov process whose state space is the set of nonnegative integers and its transition probabilities $p_{mn}(t)$

$$p_{mn}(t) := \Pr\{X(t) = n \mid X(0) = m\}, \tag{1.1}$$

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satisfy

$$p_{mn}(t) = \begin{cases} \lambda_m t + o(t), & n = m+1 \\ \mu_m t + o(t), & n = m-1 \\ 1 - (\lambda_m + \mu_n) t + o(t), & n = m, \end{cases}$$

λ_n and μ_n being the birth and death rates, respectively. It is assumed that $\lambda_n > 0$, $\mu_{n+1} > 0$ for $n \geqslant 0$ and $\mu_0 \geqslant 0$. Karlin and McGregor [13, 14] proved that

$$p_{mn}(t) = \pi_n \int_0^\infty e^{-tx} Q_m(x)\, Q_n(x)\, d\mu(x), \tag{1.2}$$

where

$$\pi_0 = 1, \qquad \pi_n = (\lambda_0 \lambda_1 \cdots \lambda_{n-1})/(\mu_1 \mu_2 \cdots \mu_n),\ n > 0, \tag{1.3}$$

and $\{Q_n(x)\}$ are polynomials orthogonal with respect to $d\mu$ and are generated by

$$Q_0(x) = 1, \qquad Q_1(x) = (\lambda_0 + \mu_0 - x)/\lambda_0 \tag{1.4}$$

$$-xQ_n(x) = \lambda_n Q_{n+1}(x) + \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)\, Q_n(x), \qquad n > 0. \tag{1.5}$$

The spectral measure $d\mu$ is normalized by being continuous on the left, $\mu(-\infty) = 0$ and the total μ mass is 1.

We found it more convenient to use the polynomials $\{F_n(x)\}$

$$F_n(x) := \pi_n Q_n(x), \tag{1.6}$$

which satisfy the orthogonality relation

$$\int_0^\infty F_n(x)\, F_m(x)\, d\mu(x) = \pi_n\, \delta_{mn}, \tag{1.7}$$

and are generated by

$$F_0(x) = 1, \qquad F_1(x) = (\lambda_0 + \mu_0 - x)/\mu_1, \tag{1.8}$$

$$-xF_n(x) = \mu_{n+1} F_{n+1}(x) + \lambda_{n-1} F_{n-1}(x) - (\lambda_n + \mu_n)\, F_n(x). \tag{1.9}$$

Given birth rates $\{\lambda_n\}$ and death rates $\{\mu_n\}$ one would like to compute, or at least say something about, the transition probabilities $\{p_{mn}(t)\}$. The Karlin–McGregor integral representation (1.2) enables us to describe $p_{mn}(t)$ when we know the spectral measure $d\mu(x)$.

In this paper we consider two linear models, namely

$$\text{Model I}: \lambda_n = n+\alpha+c+1, \qquad \mu_n = n+c, \qquad n \geqslant 0, \tag{1.10}$$

$$\text{Model II}: \lambda_n = n+\alpha+c+1, \qquad \mu_{n+1} = n+c+1, \qquad n \geqslant 0,\ \mu_0 = 0. \tag{1.11}$$

In Model I the F_n's are the associated Laguerre polynomials $\{L_n^\alpha(x; c)\}$ of Askey and Wimp [2]. The spectral measure of Model I was computed by Askey and Wimp in their aforementioned work. The spectral measures of Models I and II will be computed in Section 2. The approach adopted in this work uses generating functions and has been previously used by many authors (see, e.g., [3, 15, 19]). In Section 2 we shall outline a rigorous version of the generating function method and apply it to obtain the spectral measures of Models I and II. In Section 3 we derive an explicit formula for the polynomials $F_n(x)$ in both models and show how the spectral measures of one model can be obtained from the other one. In Section 3 we also derive an explicit formula for the numerator polynomials $\{F_n^*(x)\}$ in the continued fractions whose denominators are $\{F_n(x)\}$ of Models I and II. The connection with the associated Hermite polynomials will also be mentioned. These explicit formulas in the case of Model I were established by Askey and Wimp in [2] using a completely different approach.

So far we discussed asymptotically symmetric linear models, that is, linear models satisfying

$$\lim_{n\to\infty} \mu_n/\lambda_n = 1.$$

When $\lambda_n = a$, $\mu_n = n$, the F_n's are essentially the Charlier polynomials, [7, Sect. 10.25]. The Meixner polynomials [5, VI.3] arise when $\lambda_n = c(n+\beta)$, $\mu_n = n$, $0 < c < 1$. In Section 4 we discuss the cases

$$\lambda_n = c(n+\gamma+\beta), \qquad \mu_n = n+\gamma, \qquad n \geqslant 0, \tag{1.12}$$

$$\lambda_n = c(n+\gamma+\beta), \qquad \mu_{n+1} = n+\gamma+1, \qquad n \geqslant 0,\ \mu_0 = 0. \tag{1.13}$$

These are the associated Meixner polynomials. A confluent case gives

$$\lambda_n = a, \qquad \mu_n = n+\gamma \qquad n \geqslant 0, \tag{1.14}$$

and

$$\lambda_n = a, \qquad \mu_{n+1} = n+\gamma+1, \qquad n \geqslant 0,\ \mu_0 = 0. \tag{1.15}$$

In Section 4 we identify the spectra and the Stieltjes transforms of the spectral measures in the above cases.

The approach used in Section 4 to obtain spectral information has been used by Pollaczek in his pioneering work [16] and later in [1, 4]. References to the associated classical orthogonal polynomials may be found in [2, 4]. For the theory and many interesting applications of birth and death processes the interested reader may consult Feller's volumes, [8, 9]. This paper ends with Section 5 where we include a brief interpretation of the cases $\mu_0 = 0$, $\mu_0 \neq 0$ and mention how the largest and smallest zeros of some of the polynomials investigated change when the parameters in the corresponding birth and death rates change.

2. Generating Functions

Let $P_m(t, w)$ be a generating function of $p_{mn}(t)$, that is,

$$P_m(t, w) = \sum_{n=0}^{\infty} w^n p_{mn}(t). \tag{2.1}$$

The function $P_m(t, w)$ is an analytic function of w in $|w| \leqslant 1$ since the series $\sum_{n=0}^{\infty} p_{mn}(t)$ is a convergent series of nonnegative terms. Furthermore (1.2) and (1.6) imply

$$\pi_m P_m(t, w) = \int_0^{\infty} e^{-tx} F_m(x) F(x, w)\, d\mu(x), \tag{2.2}$$

where

$$F(x, w) := \sum_{n=0}^{\infty} w^n F_n(x). \tag{2.3}$$

We now assume that λ_n and μ_{n+1} are polynomial functions of n, $n \geqslant 0$, and

$$\tilde{\mu}_0 = \lim_{n \to 0} \mu_n. \tag{2.4}$$

The forward Chapman–Kolomogorov equations [8, 9] are

$$\dot{p}_{m,n}(t) = \mu_{n+1} p_{m,n+1}(t) + \lambda_{n-1} p_{m,n-1}(t) - (\lambda_n + \mu_n)\, p_{m,n}(t), \qquad n \geqslant 0, \tag{2.5}$$

and $\lambda_{-1} p_{m,-1}(t)$ is interpreted as zero. Multiplying (2.5) by w^n and adding the resulting equations leads to the partial differential equation

$$\frac{\partial}{\partial t} P_m(t, w) = (1 - w)\left[\frac{1}{w}\mu(\delta) - \lambda(\delta)\right] P_m(t, w) + \left[\tilde{\mu}_0 \frac{(w-1)}{w} - \mu_0\right] p_{m,0}(t) \tag{2.6}$$

$$\delta := w\frac{\partial}{\partial w}, \tag{2.7}$$

and $\lambda(\delta)$ and $\mu(\delta)$ are the polynomials λ_δ and μ_δ, respectively. The integral representations (1.2) and (2.2) establish the differential equation

$$\left[(1-w)\left\{\frac{1}{w}\mu(\delta)-\lambda(\delta)\right\}+x\right]F(x, w)=\mu_0+\tilde{\mu}_0(1-w)/w. \qquad (2.8)$$

One can think of (2.2) as the result of separating variables in the partial differential equation (2.6) and the measure $d\mu(x)$ plays the role of separation constants. The normalization $\int_0^\infty d\mu(x)=1$ and the orthogonality of the F_n's with respect to $d\mu$ give the boundary condition

$$\int_0^\infty F(x, w)\, d\mu(x)=1, \qquad (2.9)$$

which is an integral equation involving $d\mu$.

We now solve (2.8) when λ_n and μ_n are polynomials in n of degree 1 and $\lim_{n\to\infty}\lambda_n/\mu_n=1$. Define

$$\eta:=0 \text{ in Model I}, \qquad \eta:=1 \text{ in Model II}. \qquad (2.10)$$

The differential equation (2.8) becomes

$$w(1-w)^2\frac{\partial F}{\partial w}+[(1-w)\{c-(c+\alpha+1)w\}+xw]F=c(1-w)^\eta$$

whose solution is

$$F(x, w)=w^{-c}(1-w)^{-\alpha-1}\exp\left(\frac{-x}{1-w}\right)$$
$$\times\left[C+c\int_a^w(1-u)^{\eta+\alpha-1}u^{c-1}\exp\left(\frac{x}{1-u}\right)du\right] \qquad (2.11)$$

for some constants C and a, $1>a>0$. When $c\geqslant 0$ the boundary condition $F(x, 0)=1$ implies

$$F(x, w)=cw^{-c}(1-w)^{-\alpha-1}\exp\left(\frac{-xw}{1-w}\right)$$
$$\times\int_0^w u^{c-1}(1-u)^{\eta+\alpha-1}\exp\left(\frac{xu}{1-u}\right)du. \qquad (2.12)$$

Now let $u=\tau/(1+\tau)$ and

$$z:=w/(1-w). \qquad (2.13)$$

The integral representation (2.12) becomes

$$F(x, z/(1+z)) = cz^{-c}(1+z)^{c+\alpha+1} \int_0^z \tau^{c-1}(1+\tau)^{-\alpha-c-\eta} e^{x(\tau-z)}\, d\tau. \tag{2.14}$$

In the case under consideration the boundary condition (2.9) is

$$z^c(1+z)^{-c-\alpha-1} = c\int_0^\infty \left\{\int_0^z \tau^{c-1}(1+\tau)^{-\alpha-c-\eta} e^{-x(z-\tau)}\, d\tau\right\} d\mu(x).$$

The inner integral in the above equality is a convolution of two functions, so we apply the Laplace transform to the above identity and obtain the relationship

$$\int_0^\infty \frac{d\mu(x)}{x+p} = \Psi(c+1, 1-\alpha; p)/\Psi(c, 1-\alpha-\eta; p). \tag{2.15}$$

The birth and death rates λ_n and μ_{n+1} are always assumed to be positive for $n \geqslant 0$ and μ_0 is assumed to be nonnegative. This forces $c \geqslant 0$, $c+\alpha+1>0$ in Model I but only requires $c>-1$, $c+\alpha+1>0$ in Model II. When $0>c>-1$ in Model II the integral representation (2.14) is not valid but we can go back to (2.11), write cu^{c-1} as $(d/du)\, u^c$, integrate by parts, and then apply the boundary condition (2.9). The result is

$$\int_0^\infty \frac{d\mu(x)}{x+p} = \frac{\Psi(c+1, 2-\alpha; p) - (c+1)\,\Psi(c+2, 2-\alpha; p)}{\alpha\Psi(c+1, 1-\alpha; p) + p\Psi(c+1, 2-\alpha; p)}. \tag{2.16}$$

Using the contiguous relations (6.6.6) and (6.6.7) in [7, p. 258] one can reduce the right-hand side of (2.16) when $c<0$ to the right-hand side of (2.15). We now evaluate the spectral measures in Models I and II.

THEOREM 2.1. *Let $L_n^\alpha(x;c)$ and $\mathscr{L}_n^\alpha(x;c)$ be the F_n's in Models* I *and* II, *respectively, and let μ_1 and μ_2 be their spectral measures, respectively. Then*

(i) $\int_0^\infty (d\mu_j(x)/(x+p)) = \Psi(c+1, 1-\alpha; p)/\Psi(c, 2-\alpha-j; p)$, $j=1, 2$.

Furthermore μ_j, $j=1, 2$, are absolutely continuous and

(ii) $\mu_1'(x) = x^\alpha e^{-x}|\Psi(c, 1-\alpha, xe^{-i\pi})|^{-2}/\{\Gamma(c+1)\Gamma(1+c+\alpha)\}$,

(iii) $\mu_2'(x) = x^\alpha e^{-x}|\Psi(c, -\alpha; xe^{-i\pi})|^{-2}/\{\Gamma(c+1)\Gamma(1+c+\alpha)\}$,

and the polynomials $\{L_n^\alpha(x, c)\}$ and $\{\mathscr{L}_n^\alpha(x;c)\}$ satisfy the orthogonality relation

(iv) $\int_0^\infty p_{n,j}(x)\, p_{m,j}(x)\, d\mu_j(x) = ((\alpha+c+1)_n/(c+1)_n)\, \delta_{m,n}$, $j=1, 2$,

$p_{n,1}(x) = L_n^\alpha(x; c)$, $p_{n,2}(x) = \mathscr{L}_n^\alpha(x; c)$.

Proof. We already established part (i) in the discussion before the Theorem. The measures $d\mu_1$ and $d\mu_2$ can be computed from the Perron–Stieltjes inversion formula

$$F(\rho)=\int_0^\infty \frac{d\mu(x)}{x+\rho} \quad \text{iff} \quad \mu(x_2)-\mu(x_1)=\lim_{\varepsilon\to 0_+}\int_{x_1}^{x_2}\frac{F(-x-i\varepsilon)-F(-x+i\varepsilon)}{2\pi i}\,dx.$$

The details of computing μ_2 are in Ismail and Kelker [12] and μ_1 can be similarly evaluated. See also Askey and Wimp [2]. Finally the orthogonality relation (iv) follows from (ii), (iii), and (1.9). This completes the proof.

Remark. The measure μ_2 is also implicit in the work of Goovaerts, D'hooge, and DePril [10].

3. Explicit Representation

In order to find an explicit formula for $F_n(x)$ we expand the right-hand side of (2.14) in powers of z then expand $z=w/(1-w)$ in powers of w. We then identify the coefficient of w^n in $F(x,w)$ as $F_n(x)$. The first step is

$$\begin{aligned} &F(x, z/(1+z)) \\ &\quad = c(1+z)^{c+\alpha+1}\int_0^1 \tau^{c-1}(1+z\tau)^{-\alpha-c-\eta}\sum_{m=0}^{\infty}\frac{\{xz(\tau-1)\}^m}{m!}\,d\tau \\ &\quad = \Gamma(c+1)(1+z)^{c+\alpha+1}\sum_{m=0}^{\infty}\frac{(-xz)^m}{\Gamma(c+m+1)}\,{}_2F_1\left(\begin{matrix} c,\ \alpha+\eta+c \\ m+c+1\end{matrix}\,\middle|\,-z\right), \end{aligned}$$

where we used the familiar integral representation [6, 2.1.10, p. 59; 17, p. 47]

$${}_2F_1\left(\begin{matrix} A,\ B \\ C\end{matrix}\,\middle|\,z\right)=\frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)}\int_0^1 t^{B-1}(1-t)^{C-B-1}(1-zt)^{-A}\,dt. \tag{3.1}$$

We now apply the Pfaff–Kummer transformation [17, p. 60]

$${}_2F_1\left(\begin{matrix} A,\ B \\ C\end{matrix}\,\middle|\,z\right)=(1-z)^{-A}\,{}_2F_1\left(\begin{matrix} A,\ C-B \\ C\end{matrix}\,\middle|\,\frac{z}{z-1}\right) \tag{3.2}$$

and the binomial theorem to obtain

$$F(x,w)=\sum_{m=0}^{\infty}\frac{(-xw)^m}{(c+1)_m}(1-w)^{-m-\alpha-1}\,{}_2F_1\left(\begin{matrix} c,\ m+1-\alpha-\eta \\ m+c+1\end{matrix}\,\middle|\,w\right). \tag{3.3}$$

Hence

$$F(x, w) = \sum_{j,m,k=0}^{\infty} \frac{(-x)^m (\alpha+1+m)_j (c)_k (m+1-\alpha-\eta)_k}{(c+1)_m j!\,(m+c+1)_k k!} w^{m+j+k}.$$

This establishes the explicit representation

$$\begin{aligned} F_n(x) &= F_n(x; \alpha, c; n) \\ &= \frac{(\alpha+1)_n}{n!} \sum_{m=0}^{n} \frac{(-n)_m x^m}{(c+1)_m (\alpha+1)_m} {}_3F_2\left(\begin{matrix} m-n,\, m+1-\alpha-\eta,\, c \\ -\alpha-n,\, c+m+1 \end{matrix}\middle| 1\right). \end{aligned} \tag{3.4}$$

When α is an integer, one has to interpret the ${}_3F_2$ appearing in (3.4) as

$$\sum_{k=0}^{n-m} \frac{(n-m)_k (m+1-\alpha-\eta)_k (c)_k}{(-\alpha-n)_k (c+m+1)_k\, k!}.$$

The representation (3.4) when $\eta = 0$ was proved in [2] using detailed information on the two linear independent solutions of the three term recurrence relation (1.9).

The associated Hermite polynomials $H_n(x; c)$ are generated by

$$H_{n+1}(x; c) = 2xH_n(x; c) - 2(n+c)\, H_{n-1}(x; c), \qquad n > 0, \tag{3.5}$$

with $H_0(x; c) = 1$, $H_1(x; c) = 2x$.

Askey and Wimp [2] proved the orthogonality relation

$$\int_{-\infty}^{\infty} \frac{H_m(x; c)\, H_n(x; c)}{|D_{-c}(xe^{i\pi/2}\sqrt{2}|^2}\, dx = 2^n \sqrt{\pi}\, \Gamma(n+c+1)\, \delta_{m,n}, \tag{3.6}$$

where D_ν is a parabolic cylinder function [6, 7]. They also observed that

$$H_{2n+1}(x; c) = 2x\sigma_n L_n^{1/2}(x^2; c/2), \qquad \sigma_n := (-4)^n\, (1+c/2)_n, \tag{3.7}$$

$$H_{2n}(x; c) = \sigma_n \left\{ L_n^{-1/2}(x^2, c/2) - \frac{c}{c+2} L_{n-1}^{-1/2}(x^2, 1+c/2) \right\}. \tag{3.8}$$

They explained why $H_{2n}(x; c)$ cannot simply be a multiple of $L_n^{-1/2}(x^2, c)$. At the end of their paper they said, "The existence of (5.4) ((3.8) above) suggests there are more sets of orthogonal polynomials that can be found from the results of (their) Section 2." They were right. The $\mathscr{L}_n$'s seem to be all that is missing to complete the picture. When $j = 2$ part (iv) of Theorem 2.1 gives

$$\int_{-\infty}^{\infty} \frac{e^{-x^2} \mathscr{L}_n^{-1/2}(x^2; c)\, \mathscr{L}_m^{-1/2}(x^2; c)\, dx}{|\Psi(c, \frac{1}{2}; -x^2)|^2} = \frac{\Gamma(\frac{1}{2}+c+n)\Gamma(c+1)}{(c+1)_n} \delta_{m,n}.$$

Finally (6.9.31) in [6], namely

$$D_{2\nu}(2x) = 2^{\nu} e^{-x^2} \Psi(-\nu, \tfrac{1}{2}; 2x^2),$$

the duplication formula $\sqrt{\pi}\, \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z+\frac{1}{2})$ [17, Chap. 2], and the uniqueness of the orthogonal polynomials establish

$$H_{2n}(x; c) = \sigma_n \mathscr{L}_n^{-1/2}(x^2; c/2). \tag{3.9}$$

The identity (3.9) can also be proved using (3.5) and (1.9).

4. The Associated Meixner and Charlier Polynomials

Set

$$\eta = 0 \text{ in case (1.12)}, \qquad \eta = 1, \text{ in case (1.13)} \tag{4.1}$$

and let

$$M_n^{\gamma}(x; B, c, \eta) \text{ be } F_n(x/(1-c)), \qquad \eta = 0, 1. \tag{4.2}$$

The generating functions

$$F_{\eta}(x, w) = \sum_{n=0}^{\infty} w^n M_n^{\gamma}(x; \beta, c, \eta) \tag{4.3}$$

are given by

$$F_{\eta}(x, w) = \gamma w^{-\gamma} (1 - cw)^{-\beta - x} (1 - w)^x$$

$$\times \int_0^w u^{\gamma - 1} (1 - cu)^{\beta + x - 1} (1 - u)^{\eta - x - 1}\, du. \tag{4.4}$$

Applying Darboux's method (in Szegö, [21, Sect. 8.4]), to (4.4) gives

$$n^{x+1} M_n^{\gamma}(x; \beta, c, \eta) \sim (1-c)^{-\beta - x} \frac{\Gamma(\gamma+1)\Gamma(\eta - x)}{\Gamma(\gamma + \eta - x)\Gamma(-x)} {}_2F_1\left(\left.\begin{matrix}\gamma, 1-\beta-x \\ \eta+\gamma-x\end{matrix}\right| c\right). \tag{4.5}$$

The numerators of the continued fraction associated with either set of polynomials are $-(\gamma+1)^{-1} M_n^{\gamma+1}(x; \beta, c, 0)$. The β_n's in the notation of Shohat and Tamarkin [20] are $\lambda_{n-1}\mu_n$, so $\sum \beta_n^{-1/2}$ diverges and a criterion of Carleman [20, p. 59] establishes the determinancy of the moment problem in both cases ($\eta = 0, 1$), that is, both spectra measures are essentially unique. Let $\{M_n^\gamma(x; \beta, c, n)\}$ be orthogonal with respect to a measure $d\zeta_\eta(x)$. Therefore we obtain

$$\int_0^\infty \frac{d\zeta_\eta(u)}{x+u} = (\gamma+x)^{1-\eta}\,(x)^{-\eta}\,{}_2F_1\left(\left.\begin{matrix}1+\gamma,\, 1+x-\beta\\ \gamma+1+x\end{matrix}\right| c\right)\Big/{}_2F_1\left(\left.\begin{matrix}\gamma,\, 1+x-\beta\\ \eta+\gamma+x\end{matrix}\right| c\right) \tag{4.6}$$

from (4.5) and Theorem 2.9, p. 50 in [20]. The relationship (4.6) proves that the spectrum of both processes is discrete and is located at the zeros of

$$\frac{\Gamma(\eta-x)}{\Gamma(-x)\Gamma(\gamma+\eta-x)}\,{}_2F_1\left(\left.\begin{matrix}\gamma,\, 1-\beta-x\\ \eta+\gamma-x\end{matrix}\right| c\right). \tag{4.7}$$

Similarly we let $\eta=0$ (or 1) in the case (1.14) (or (1.15)). The spectrum of the corresponding process consists of the zeros of

$$\frac{\Gamma(\eta-x)}{\Gamma(-x)\Gamma(\gamma+\eta-x)}\,{}_1F_1\left(\left.\begin{matrix}\gamma\\ \gamma+\eta-x\end{matrix}\right| -a\right). \tag{4.8}$$

If $d\xi_\eta$ are the spectral measures of the corresponding processes then

$$\int_0^\infty \frac{d\xi_\eta(u)}{x+u} = (\gamma+x)^{1-\eta}\,x^{-\eta}\,{}_1F_1\left(\left.\begin{matrix}\gamma+1\\ \gamma+1+x\end{matrix}\right| -a\right)\Big/{}_1F_1\left(\left.\begin{matrix}\gamma\\ \eta+\gamma+x\end{matrix}\right| -a\right). \tag{4.9}$$

The associated Charlier polynomials are orthogonal with respect to the discrete measure $d\xi_\eta(u)$.

5. Concluding Remarks

In the models discussed in this work we distinguished between the cases $\mu_0=0$ and $\mu_0>0$. One may think of the states of a birth and death process as the sizes of a population or the positions of a particle. In immigration or migration models [3], μ_0 must vanish because nobody dies in, or leaves, a community of zero population. On the other hand models with $\mu_0>0$ have an absorbing barrier at -1 and once a particle reaches the state -1 it stays there forever after. Birth and death processes with $\mu_0>0$ are always not "honest" (Reuter [18]), since $\sum_{n=0}^\infty p_{mn}(t)<1$. One may think of the case $\mu_0>0$ to model a bank account where a state or a population size represents the balance in the account. In such a model the effect of $\mu_0>0$ is to freeze the account once the balance reaches -1.

It is worth pointing out that, in general, a birth and death process with $\tilde{\mu}_0(=\lim_{n\to 0}\mu_n)\neq 0$ gives rise to two distinct families of birth and death process polynomials. The two families correspond to the choices $\mu_0 = 0$ and $\mu_0 = \tilde{\mu}_0$.

We now briefly discuss the monotonicity of the largest and smallest zeros of the polynomials considered in this paper. The Perron–Frobenius theorem [22, Chap. 3] or [11] shows that the largest zero of all the polynomials considered in this paper are strictly increasing functions of the parameters involved. Theorem 2 in [11] asserts that the smallest zero of a birth and death process polynomial $Q_N(x)$ when $\mu_0 = 0$ increases (decreases) with a parameter if b_0 and both b_n and b_n/d_n, $N > n > 0$, are increasing (decreasing) functions of the same parameter. This type of question, that is the monotonicity of zeros, is usually answered using either a Sturmian argument [21] or Markov's theorem [21, Sect. 6.21]. In all the cases considered in this work neither approach is applicable.

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